(2). Let FG be the random line, $AH=x$, $DG=y$.
Then $FH=\frac{1}{3}\sqrt{3}x$, $FG=\sqrt{[(a\sqrt{3}-x)^2+(y-\frac{1}{3}\sqrt{3}x)^2}=l$.

$$\triangle=\text{average length}=\frac{\int_0^{a\sqrt{3}}\int_0^a l\,dx\,dy}{\int_0^{a\sqrt{3}}\int_0^a dx\,dy}$$

$$\triangle=\frac{1}{a^2\sqrt{3}}\int_0^{a\sqrt{3}}\int_0^a \sqrt{[(a\sqrt{3}-x)^2+(y-\tfrac{1}{3}\sqrt{3}x)^2]}\,dx\,dy$$

$$=\frac{1}{12a^2\sqrt{3}}\int_0^{a\sqrt{3}}\Big[4(a\sqrt{3}-x)^2+x\sqrt{[9a^2+(4x-3\sqrt{3}\,a)^2]}$$

$$-6(a\sqrt{3}-x)^2\log\left(\frac{\sqrt{[9a^2+(4x-3\sqrt{3}\,a)^2]}-2x}{6(\sqrt{3}-x)}\right)\Big]dx.$$

$\therefore\ \triangle=(a/16)[6+2\sqrt{3}+4\log 3+\log(3+2\sqrt{3})]$.

82. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Find the average area of the quadrilateral formed by joining the extremities of two chords perpendicular to each other and passing through a point at a distance a from the center of a circle radius R.

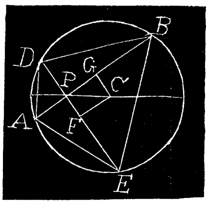
Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.; and the PROPOSER.

Let $\triangle$=average area, $\angle BPC=\theta$, P the given point, C the center of the cirle, $CP=a$, $CB=R$.

CASE I. $a<R$, $CG=a\sin\theta$, $CF=a\cos\theta$.

$\therefore\ AB=2\sqrt{(R^2-a^2\sin^2\theta)}$, $DE=2\sqrt{(R^2-a^2\cos^2\theta)}$.

Area $ADBE=2\sqrt{[(R^2-a^2\sin^2\theta)(R^2-a^2\cos^2\theta)]}$
$=2R^2\sqrt{[(1-e^2\sin^2\theta)(1-e^2\cos^2\theta)]}$ where $e^2=a^2/R^2$.



$$\therefore\ \triangle=2R^2\int_0^{\frac{1}{2}\pi}\sqrt{[(1-e^2\sin^2\theta)(1-e^2\cos^2\theta)]}\,d\theta\Big/\int_0^{\frac{1}{2}\pi}d\theta$$

$$=\frac{2R^2}{\pi}(2-e^2)\,E\left(\frac{e^2}{2-e^2},\ \tfrac{1}{2}\pi\right).$$

(See page 356, No. 10, Vol. I, for above integration.

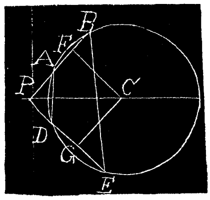
CASE II. $a>R$ and $a<R\sqrt{2}$, $PF=a\cos\theta$, $PG=a\sin\theta$;
$PB=a\cos\theta+\sqrt{(R^2-a^2\sin^2\theta)}$, $PA=a\cos\theta-\sqrt{(R^2-a^2\sin^2\theta)}$.
$PE=a\sin\theta+\sqrt{(R^2-a^2\sin^2\theta)}$, $PD=a\sin\theta-\sqrt{(R^2-a^2\cos^2\theta)}$.
Area $ABED=\frac{1}{2}(BP.PE-AP.PD)$
$$=a[\sqrt{(R^2-a^2\sin^2\theta)}\sin\theta+\sqrt{(R^2-a^2\cos^2\theta)}\cos\theta].$$
The limits of θ are $\frac{1}{2}\pi-\sin^{-1}(R/a)=\theta''$ and $\sin^{-1}(R/a)=\theta'$.

$$\therefore\ \triangle=aR\int_{\theta''}^{\theta'}[\surd(1-e^2\sin^2\theta)\sin\theta+\surd(1-e^2\cos^2\theta)\cos\theta]d\theta\Big/\int_{\theta''}^{\theta'}d\theta$$

$$=\frac{aR}{4\sin^{-1}(R/a)-\pi}\Big[\sin\theta'\surd(1-e^2\cos^2\theta')$$

$$-\sin\theta''\surd(1-e^2\cos^2\theta'')-\cos\theta'\surd(1-e^2\sin^2\theta')$$

$$+\cos\theta''\surd(1-e^2\sin^2\theta'')$$



$$+\frac{1-e^2}{e}\log\left(\frac{e\sin\theta'+\surd(1-e^2\cos^2\theta')}{e\sin\theta''+\surd(1-e^2\cos^2\theta'')}\right)$$

$$-\frac{1-e^2}{e}\log\left(\frac{e\cos\theta'+\surd(1-e^2\sin^2\theta')}{e\cos\theta''+\surd(1-e^2\sin^2\theta'')}\right)\Big].$$

But $\sin\theta'=\cos\theta''=R/a$, $\sin\theta''=\cos\theta'=[\surd(a^2-R^2)]/a$.
$e^2\sin^2\theta'=e^2\cos^2\theta''=1$, $e^2\sin^2\theta''=e^2\cos^2\theta'=[(a^2-R^2)/R^2]=e^2-1$.
Whence by substitution and reduction we get

$$\triangle=\frac{R^2}{2\sin^{-1}(R/a)-\frac{1}{2}\pi}\Big[\surd(2-e^2)+(1-e^2)\log\Big(\frac{1+\surd(2-e^2)}{\surd(e^2-1)}\Big)\Big].$$

83. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy, Irving College, Mechanicsburg, Pa.

Find the average area of all ellipses whose semi-axis major is a.

I. Solution by J. W. YOUNG, Fellow and Assistant, Ohio State University, Columbus, Ohio.

The area of an ellipse whose major-axis is a, and whose minor-axis is b, is πab. We must find the average of all possible values of this expression as b varies from zero to a.

$$\therefore\ \text{Average required}=\frac{\pi a\int_0^a bdb}{\int_0^a db}=\tfrac{1}{2}\pi a^2,$$

$=\frac{1}{2}$ the area of the circle whose radius is the major-axis of the ellipse.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

1. Let x=semi-conjugate axis. Then average area

$$=\pi a\frac{\int_0^a xdx}{\int_0^a dx}=\tfrac{1}{2}\pi a^2=1.5708a^2.$$